Mathematica Moravica
Vol. **9** (2005), 77–86

About a Class of $(n,m)-$Goups

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Abstract. In this paper $(km,m)-$groups, $k \geq 3$, with one condition are described.

1. Preliminaries

Definition 1.1 ([1]). Let $n \geq m+1$ and let $(Q;A)$ be an $(n,m)-$groupoid $(A: Q^n \to Q^m;\ n,m \in N)$. We say that $(Q;A)$ is an $(n,m)-$**group** iff the following statements hold:

(I) For every $i,j \in \{1,\dots,n-m+1\},\ i<j$, the following law holds
$$A(x_1^{i-1}, A(x_i^{i+n-1}), x_{i+n}^{2n-m}) = A(x_1^{j-1}, A(x_j^{j+n-1}), x_{j+n}^{2n-m})$$
[$:< i,j > -$associative law][1]; and

(II) For every $i \in \{1,\dots,n-m+1\}$ and for every $a_1^n \in Q$ there is exactly one $x_1^m \in Q^m$ such that the following equality holds
$$A(a_1^{i-1}, x_1^m, a_i^{n-m}) = a_{n-m+1}^n.$$
See, also [3].

Definition 1.2 ([6]). Let $n \geq 2m$ and let $(Q;A)$ be a $(n,m)-$groupoid. Let also $\mathbf{e}$ be a mapping of the set Q^{n-2m} into the set Q^m. Then, we say that $\mathbf{e}$ is an $\{1,n-m+1\}-$**neutral operation of the** $(n,m)-$**groupoid** $(Q;A)$ iff for every sequence a_1^{n-2m} over Q and for every $x_1^m \in Q^m$ the following equalities hold
$$A(x_1^m, a_1^{n-2m}, \mathbf{e}(a_1^{n-2m})) = x_1^m$$
and
$$A(\mathbf{e}(a_1^{n-2m}), a_1^{n-2m}, x_1^m) = x_1^m.$$

Remark 1.1. For $m=1$ $\mathbf{e}$ is an $\{1,n\}-$neutral operation of the $n-$groupoid $(Q;A)$ [5]. Cf. Chapter II in [9].

Proposition 1.1 ([6]). *Let $(Q;A)$ be an $(n,m)-$groupoid and let $n \geq 2m$. Then there is at most one $\{1,n-m+1\}-$neutral operation of $(Q;A)$.*

2000 *Mathematics Subject Classification.* Primary: 20N15.
Key words and phrases. $(n,m)-$group, $\{1,n-m+1\}-$neutral operation of the $(n,m)-$groupoid.

[1]1) $(Q;A)$ is an $(n,m)-$semigroup

Proposition 1.2 ([6]). *Every $(n,m)-$group $(n \geq 2m)$ has an $\{1, n-m+1\}-$neutral operation.*

See, also [8].

Proposition 1.3 ([8]). *Let $n \geq 2m$ and let $(Q;A)$ be an $(n,m)-$groupoid. Further on, let the $< 1, n-m+1 > -$associative law holds in $(Q;A)$ and for every $a_1^n \in Q$ there is at least one $x_1^m \in Q^m$ and at least one $y_1^m \in Q$ such that the following equalities hold*

$$A(a_1^{n-m}, x_1^m) = a_{n-m+1}^n$$

and

$$A(y_1^m, a_1^{n-m}) = a_{n-m+1}^n.$$

Then there are mappings $\mathbf{e}$ *and* $^{-1}$*, respectively, of the sets Q^{n-2m} and Q^{n-m} into the set Q^m such that the following laws hold in the algebra $(Q; A, ^{-1}, \mathbf{e})$:*

$$A(\mathbf{e}(a_1^{n-2m}), a_1^{n-2m}, x_1^m) = x_1^m,$$
$$A(x_1^m, a_1^{n-2m}, \mathbf{e}(a_1^{n-2m})) = x_1^m,$$
$$A((a_1^{n-2m}, x_1^m)^{-1}, a_1^{n-2m}, x_1^m) = \mathbf{e}(a_1^{n-2m}),$$
$$A(x_1^m, a_1^{n-2m}, (a_1^{n-2m}, x_1^m)^{-1}) = \mathbf{e}(a_1^{n-2m}).$$

(Cf. 1.2-1.4)

2. Auxiliary proposition

Proposition 2.1 ([8]). *Let $n > m+1$ and let $(Q;A)$ be an $(n,m)-$groupoid. Also let*

(*a*) *$< 1, 2 >$-associative law hold in $(Q;A)$; and*
(*b*) *For every $x_1^m, y_1^m, a_1^{n-m} \in Q$ the following implication holds*

$$A(x_1^m, a_1^{n-m}) = A(y_1^m, a_1^{n-m}) \Rightarrow x_1^m = y_1^m.$$

Then $(Q;A)$ is an $(n,m)-$semigroup.

Proposition 2.2 ([3]). *Let $(Q;A)$ be an $(n,m)-$groupoid and $n \geq m+2$. Also, let the following statements hold: 1) $(Q;A)$ is an $(n,m)-$semigroup; 2) For every $a_1^n \in Q$ there is exactly one $x_1^m \in Q^m$ such that the following equality holds $A(a_1^{n-m}, x_1^m) = a_{n-m+1}^n$; and 3) For every $a_1^n \in Q$ there is exactly one $y_1^m \in Q^m$ such that the following equality holds $A(y_1^m, a_1^{n-m}) = a_{n-m+1}^n$. Then $(Q;A)$ is an $(n,m)-$group.*

Definition 2.1. Let $(Q;B)$ be a $(2m,m)-$groupoid and $m \geq 2$. Then: (α) $\overset{1}{B} \overset{def}{=} B$; and (β) for every $s \in N$ and for every $x_1^{(s+2)m} \in Q$

$$\overset{s+1}{B}(x_1^{(s+2)m}) \overset{def}{=} B(\overset{s}{B}(x_1^{(s+1)m}), x_{(s+1)m+1}^{(s+2)m}).$$

Proposition 2.3. *Let $(Q;B)$ be a $(2m,m)-$semigroup, $m \geq 2$ and $s \in N$. Then, for every $x_1^{(s+2)m} \in Q$ and for every $t \in \{1,\ldots,sm+1\}$ the following equality holds*

$$\overset{s+1}{B}(x_1^{(s+2)m}) = \overset{s}{B}(x_1^{t-1}, B(x_t^{t+2m-1}), x_{t+2m}^{(s+2)m}).$$

Sketch of the proof.

1) $s=1$: By Def. 1.1 and by Def. 2.3, we have

$$\overset{2}{B}(x_1^{3m}) = B(x_1^{i-1}, B(x_i^{i+2m-1}), x_{i+2m}^{3m})$$

for every $x_1^{3m} \in Q$ and for all $i \in \{1,\ldots,m+1\}$.

2) $s=v$: Let for every $x_1^{(s+2)m} \in Q$ and for all $t \in \{1,\ldots,vm+1\}$ the following equality holds

$$\overset{v+1}{B}(x_1^{(s+2)m}) = \overset{v}{B}(x_1^{t-1}, B(x_t^{t+2m-1}), x_{t+2m}^{(v+2)m}).$$

3) $v \to v+1$:

$$\overset{(v+1)+1}{B}(x_1^{(v+3)m}) \overset{(\beta)}{=} B(\overset{v+1}{B}(x_1^{(v+2)m}), x_{(v+2)m+1}^{(v+3)m}) \overset{2)}{=}$$

$$B(\overset{v}{B}(x_1^{t-1}, B(x_t^{t+2m-1}), x_{t+2m}^{(v+2)m}), x_{(v+2)m+1}^{(v+3)m})\overset{(}{\beta})=$$

$$\overset{v+1}{B}(x_1^{t-1}, B(x_t^{t+2m-1}), x_{t+2m}^{(v+3)m}) \overset{2)}{=}$$

$$\overset{v}{B}(x_1^{t-1}, B(B(x_t^{t+2m-1}), x_{t+2m}^{t+3m-1}), x_{t+3m}^{(v+3)m}) \overset{1.1}{=}$$

$$\overset{v}{B}(x_1^{t-1}, B(x_t^{t+i-2}, B(x_{t+i-1}^{t+i+2m-2}), x_{t+i+2m-1}^{t+3m-1}), x_{t+3m}^{(v+3)m}) \overset{2)}{=}$$

$$\overset{v+1}{B}(x_1^{t-1}, x_t^{t+i-2}, B(x_{t+i-1}^{t+i+2m-2}), x_{t+i+2m-1}^{t+3m-1}, x_{t+3m}^{(v+3)m}) =$$

$$\overset{v+1}{B}(x_1^{t+i-2}, B(x_{t+i-1}^{t+i+2m-2}), x_{t+i+2m-1}^{(v+3)m}).$$

□

By Def. 1.1, Def. 2.3 and by Prop. 2.4, we obtain:

Proposition 2.4. *Let $(Q;B)$ be a $(2m,m)-$semigroup, $m \geq 2$ and $(i,j) \in N^2$. Then, for every $x_1^{(i+j+1)m} \in Q$ and for all $t \in \{1,\ldots,im+1\}$ the following equality holds*

$$\overset{i+j}{B}(x_1^{(i+j+1)m}) = \overset{i}{B}(x_1^{t-1}, \overset{j}{B}(x_t^{t+(j+1)m-1}), x_{t+(j+1)m}^{(i+j+1)m}).$$

By 1.3 and by 1.4, we have:

Proposition 2.5 ([2])**.** *Let $(Q;B)$ be an $(n,m)-$group and $n=2m$. Then there is exactly one $e_1^m \in Q^m$ such that for all $x_1^m \in Q^m$ the following equalities hold*

$$(n) \qquad B(x_1^m, e_1^m) = x_1^m \quad \textit{and} \quad B(e_1^m, x_1^m) = x_1^m.$$

Remark 2.1. For $m=1$, e_1^m is a neutral element of the group $(Q;B)$.

Proposition 2.6 ([2]). *Let $(Q;B)$ be a $(2m,m)-$group, and let $e_1^m \in Q^m$ satisfying (n) [from 2.6] for all $x_1^m \in Q^m$. Then for all $i \in \{0,1,\dots,m\}$ and for every $x_1^m \in Q^m$ the following equality holds*

$$B(x_1^i, e_1^m, x_{i+1}^m) = x_1^m.$$

Sketch of the proof. $m > 1$:

$$\begin{aligned} B(x_1^i, e_1^m, x_{i+1}^m) &\overset{(n)}{=} B(e_1^m, A(x_1^i, e_1^m, x_{i+1}^m)) \\ &\overset{1.1(I)}{=} B(e_1^i, B(e_{i+1}^m, x_1^i, e_1^m), x_{i+1}^m) \\ &\overset{(n)}{=} B(e_1^i, e_{i+1}^m, x_1^i, x_{i+1}^m) \\ &= B(e_1^m, x_1^m) \overset{(n)}{=} x_1^m. \end{aligned}$$

□

Proposition 2.7 ([2]). *Let $(Q;B)$ be a $(2m,m)-$group, and let $e_1^m \in Q^m$ satisfying (n) [from 2.6] for all $x_1^m \in Q^m$. Then: $e_1 = e_2 = \dots = e_m$.*

Sketch of the proof. $m > 1$:

$$B(e_2^m, e_1^m, e_1) \overset{2.7}{=} e_2^m, e_1 \Rightarrow$$

$$B(e_2^m, e_1, e_2^m, e_1) = e_2^m, e_1 \overset{(n)}{\Rightarrow}$$

$$B(e_2^m, e_1, e_2^m, e_1) = B(e_2^m, e_1, e_1^m) \overset{1.1(II)}{\Rightarrow} e_2^m, e_1 = e_1^m,$$

whence, we obtain $e_1 = e_2 = \dots = e_m$. □

See, also [4].

3. Results

Theorem 3.1. *Let $k > 2$, $m \geq 2, n = k \cdot m$, $(Q;A)$ $(n,m)-$group and $\mathbf{e}$ its $\{1, n-m+1\}-$neutral operation. Also let exist sequence a_1^{n-2m} over Q such that for all $i \in \{0,1,\dots,2m-1\}$, and for every $x_1^{2m} \in Q$ the following equality holds*

$$A(x_1^i, a_1^{n-2m}, x_{i+1}^{2m}) = A(x_1^{2m}, a_1^{n-2m}). \tag{0}$$

Further on, let

$$B(x_1^{2m}) \overset{def}{=} A(x_1^m, a_1^{n-2m}, x_{m+1}^{2m}) \tag{1}$$

and

$$c_1^m \overset{def}{=} A\Big(\overline{\mathbf{e}(a_1^{n-2m})}^{\,k}\Big) \tag{2}$$

for all $x_1^{2m} \in Q$. Then the following statements hold

(*i*) *$(Q;B)$ is a $(2m,m)-$group;*

(*ii*) *For all* $x_1^{k\cdot m} \in Q$

$$A(x_1^{k\cdot m}) = \overset{k}{B}(x_1^{k\cdot m}, c_1^m);$$

and

(*iii*) *For all* $j \in \{0, \dots, m-1\}$ *and for every* $x_1^m \in Q$ *the following equality holds*

$$B(x_1^j, c_1^m, x_{j+1}^m) = B(x_1^m, c_1^m).$$

Proof. Firstly we prove that under the assumption the following statements hold:

1° For all $x_1^{3m} \in Q$ the following equality holds

$$B(B(x_1^{2m}), x_{2m+1}^{3m}) = B(x_1, B(x_2^{2m+1}), x_{2m+2}^{3m}).$$

2° For all $b_1^{2m} \in Q$ there is exactly one $x_1^m \in Q^m$ such that the following equality holds

$$B(x_1^m, b_1^m) = b_{m+1}^{2m}.$$

3° $(Q; B)$ is a $(2m, m)-$semigroup.

4° For all $b_1^{2m} \in Q$ there is exactly one $y_1^m \in Q^m$ such that the following equality holds

$$B(b_1^m, y_1^m) = b_{m+1}^{2m}.$$

Sketch of the proof of 1°*:*

$$\begin{aligned} B(B(x_1^{2m}), x_{2m+1}^{3m}) &\overset{(1)}{=} A(A(x_1^m, a_1^{n-2m}, x_{m+1}^{2m}), a_1^{n-2m}, x_{2m+1}^{3m}) \overset{(0)}{=} \\ &= A(A(x_1^m, a_1^{n-2m}, x_{m+1}^{2m}), x_{2m+1}, a_1^{n-2m}, x_{2m+2}^{3m}) \overset{1.1(I)}{=} \\ &= A(x_1, A(x_2^m, a_1^{n-2m}, x_{m+1}^{2m}, x_{2m+1}), a_1^{n-2m}, x_{2m+2}^{3m}) \overset{(0)(1)}{=} \\ &= B(x_1, B(x_2^{2m+1}), x_{2m+2}^{3m}). \end{aligned}$$

Sketch of the proof of 2°*:*

$$B(x_1^m, b_1^m) = b_{m+1}^{2m} \overset{(1)}{\Leftrightarrow} A(x_1^m, a_1^{n-2m}, b_1^m) = b_{m+1}^{2m},$$

whence, by Def. 1.1-(II), we obtain 2°.

Sketch of the proof of 3°*:* By 2° and by Prop. 2.1.

Sketch of the proof of 4°*:*

$$B(b_1^m, x_1^m) = b_{m+1}^{2m} \overset{(1)}{\Leftrightarrow} A(b_1^m, a_1^{n-2m}, y_1^m) = b_{m+1}^{2m},$$

whence, by Def. 1.1-(II), we have 4°.

The proof of (*i*)*:* By 2°, 3°, 4° and by Prop.2.2.

Sketch of the proof of (*ii*) *[to the case* $k = 4$*]:*

$$\begin{aligned} A(x_1^m, y_1^m, z_1^m, u_1^m) &\overset{1.4}{=} A(x_1^m, y_1^m, z_1^m, A(u_1^m, a_1^{n-2m}, \mathbf{e}(a_1^{n-2m}))) \overset{1.1(I)}{=} \\ &= A(x_1^m, y_1^m, A(z_1^m, u_1^m, a_1^{n-2m}), \mathbf{e}(a_1^{n-2m})) \overset{(0)}{=} \\ &= A(x_1^m, y_1^m, A(z_1^m, a_1^{n-2m}, u_1^m), \mathbf{e}(a_1^{n-2m})) \overset{(1)}{=} \end{aligned}$$

$$= A(x_1^m, y_1^m, B(z_1^m, u_1^m), \mathbf{e}(a_1^{n-2m})) \overset{1.4}{=}$$

$$= A(x_1^m, y_1^m, A(B(z_1^m, u_1^m), a_1^{n-2m}, \mathbf{e}(a_1^{n-2m})), \mathbf{e}(a_1^{n-2m})) \overset{1.1(I)}{=}$$

$$= A(x_1^m, A(y_1^m, B(z_1^m, u_1^m), a_1^{n-2m}), \mathbf{e}(a_1^{n-2m}), \mathbf{e}(a_1^{n-2m})) \overset{(0)}{=}$$

$$= A(x_1^m, A(y_1^m, a_1^{n-2m}, B(z_1^m, u_1^m)), \mathbf{e}(a_1^{n-2m}), \mathbf{e}(a_1^{n-2m})) \overset{(1)}{=}$$

$$= A(x_1^m, B(y_1^m, B(z_1^m, u_1^m)), \mathbf{e}(a_1^{n-2m}), \mathbf{e}(a_1^{n-2m})) \overset{1.4}{=}$$

$$= A(x_1^m, A(B(y_1^m, B(z_1^m, u_1^m)), a_1^{n-2m}, \mathbf{e}(a_1^{n-2m})), \mathbf{e}(a_1^{n-2m}), \mathbf{e}(a_1^{n-2m})) \overset{1.1(I)}{=}$$

$$= A(A(x_1^m, B(y_1^m, B(z_1^m, u_1^m)), a_1^{n-2m}), \mathbf{e}(a_1^{n-2m}), \mathbf{e}(a_1^{n-2m}), \mathbf{e}(a_1^{n-2m})) \overset{(0)}{=}$$

$$= A(A(x_1^m, a_1^{n-2m}, B(y_1^m, B(z_1^m, u_1^m))), \mathbf{e}(a_1^{n-2m}), \mathbf{e}(a_1^{n-2m}), \mathbf{e}(a_1^{n-2m})) \overset{(1)}{=}$$

$$= A\Big(B(x_1^m, B(y_1^m, B(z_1^m, u_1^m)), \mathbf{e}(a_1^{n-2m}), \overset{2}{\overline{\mathbf{e}(a_1^{n-2m})}}\Big) \overset{2.4}{=}$$

$$= A\Big(\overset{3}{B}(x_1^m, y_1^m, z_1^m, u_1^m)), A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, \mathbf{e}(a_1^{n-2m})), \overset{2}{\overline{\mathbf{e}(a_1^{n-2m})}}\Big) \overset{(0)}{=}$$

$$= A\Big(\overset{3}{B}(x_1^m, y_1^m, z_1^m, u_1^m), A(a_1^{n-2m}, \mathbf{e}(a_1^{n-2m}), \mathbf{e}(a_1^{n-2m})), \overset{2}{\overline{\mathbf{e}(a_1^{n-2m})}}\Big) \overset{1.1(I)}{=}$$

$$= A\Big(\overset{3}{B}(x_1^m, y_1^m, z_1^m, u_1^m), a_1^{n-2m}, A\Big(\overset{4}{\overline{\mathbf{e}(a_1^{n-2m})}}\Big)\Big) \overset{(1)}{=}$$

$$= B\Big(\overset{3}{B}(x_1^m, y_1^m, z_1^m, u_1^m), A\Big(\overset{4}{\overline{\mathbf{e}(a_1^{n-2m})}}\Big)\Big) \overset{(2)}{=}$$

$$= B(\overset{3}{B}(x_1^m, y_1^m, z_1^m, u_1^m), c_1^m) \overset{2.3}{=}$$

$$= \overset{4}{B}(x_1^m, y_1^m, z_1^m, u_1^m, c_1^m).$$

Sketch of a part of the proof of (iii):

By (ii) and by

$$A(A(x_1^{k\cdot m}), x_{k\cdot m+1}^{2km-m}) = A(x_1, A(x_2^{k\cdot m+1}), x_{k\cdot m+2}^{2km-m}),$$

we have

$$\overset{k}{B}(x_1, \overset{k}{B}(x_2^{k\cdot m}, c_1^m, x_{k\cdot m+1}), x_{k\cdot m+2}^{2km-m}, c_1^m) = \overset{k}{B}(x_1, \overset{k}{B}(x_2^{k\cdot m}, c_1^m), x_{k\cdot m+2}^{2km-m}, c_1^m),$$

and by Def.1.1-(II), we have

$$\overset{k}{B}(x_2^{k\cdot m}, c_1^m, x_{k\cdot m+1}) = \overset{k}{B}(x_2^{k\cdot m+1}, c_1^m),$$

i.e., by Prop. 2.4,

$$\overset{k-1}{B}(x_2^{(k-1)\cdot m+1}, B(x_{(k-1)\cdot m+2}^{k\cdot m}, c_1^m, x_{k\cdot m+1})) = \overset{k-1}{B}(x_2^{(k-1)\cdot m+1}, B(x_{(k-1)\cdot m+2}^{k\cdot m+1}, c_1^m)).$$

Finaly, hence we obtain

$$B(x_{(k-1)\cdot m+2}^{k\cdot m}, c_1^m, x_{k\cdot m+1}) = B(x_{(k-1)\cdot m+2}^{k\cdot m+1}, c_1^m),$$

i.e., we obtain (iii) for $j = m-1$. □

Theorem 3.2. *Let $m \geq 2$, $(Q;B)$ be a $(2m,m)-$group, and let $\overset{m}{e} \in Q^m$ its neutral element (cf. Prop. 2.8). Also let c_1^m be an element of the set Q^m such that for every $i \in \{0,1,\ldots,m-1\}$ and for every $x_1^m \in Q^m$ the following equality holds*

(a) $B(x_1^i, c_1^m, x_{i+1}^m) = B(x_1^m, c_1^m)$
(cf. Prop. 2.6 and Prop. 2.7). Further on, let $k > 2$ and

(b) $A(x_1^{k\cdot m}) = \overset{k}{B}(x_1^{k\cdot m}, c_1^m)$
for all $x_1^{k\cdot m} \in Q$. Then $(Q;A)$ is a $(k\cdot m, m)-$group with condition:

(c) *Exist sequence $a_1^{(k-2)\cdot m}$ over Q such that for all $j \in \{0,\ldots,2m-1\}$ and for every $x_1^{2m} \in Q$ the following equality holds*

$$A(x_1^j, a_1^{(k-2)\cdot m}, x_{j+1}^{2m}) = A(x_1^{2m}, a_1^{(k-2)\cdot m}).$$

Proof. Firstly we prove that under the assumptions the following statements hold:

$\overset{\circ}{1}$ For all $x_1^{2km-m} \in Q$ the following equality holds

$$A(A(x_1^{k\cdot m}), x_{k\cdot m+1}^{2km-m}) = A(x_1, A(x_2^{k\cdot m+1}), x_{k\cdot m+2}^{2km-m})$$

[$<1,2>$ $-$associative law*]*.

$\overset{\circ}{2}$ For all $b_1^{2km} \in Q$ there is exactly one $x_1^m \in Q^m$ such that the following equality holds

$$A(x_1^m, b_1^{k\cdot m-m}) = b_{k\cdot m-m+1}^{k\cdot m}.$$

$\overset{\circ}{3}$ $(Q;A)$ is a $(km,m)-$semigroup.

$\overset{\circ}{4}$ For all $b_1^{2km} \in Q$ there is exactly one $y_1^m \in Q^m$ such that the following equality holds

$$A(b_1^{k\cdot m-m}, y_1^m) = b_{k\cdot m-m+1}^{k\cdot m}.$$

$\overset{\circ}{5}$ For all $j \in \{0,\ldots,2m-1\}$ and for every $x_1^{2km} \in Q$ the following equality holds

$$A(x_1^j, \overset{(k-3)\cdot m}{e}, (c_1^m)^{-1}, x_{j+1}^{2m}) = A(x_1^{2m}, \overset{(k-3)\cdot m}{e}, (c_1^m)^{-1}),$$

where

(d) $B((c_1^m)^{-1}, c_1^m) = \overset{m}{e}$ *[*cf. Prop. 1.5 and Prop. 2.8*]*.

Sketch of the proof of $\overset{\circ}{1}$:

$$A(A(x_1^{k\cdot m}), x_{k\cdot m+1}^{2km-m}) \overset{(b)}{=} \overset{k}{B}(\overset{k}{B}(x_1^{k\cdot m}, c_1^m), x_{k\cdot m+1}^{2km-m}, c_1^m) \overset{2.5}{=}$$

$$= \overset{k}{B}(x_1, \overset{k}{B}(x_2^{k\cdot m}, c_1^m, x_{k\cdot m+1}), x_{k\cdot m+2}^{2km-m}, c_1^m) \overset{2.4}{=}$$

$$= \overset{k}{B}(x_1, \overset{k-1}{B}(x_2^{(k-1)\cdot m+1}, B(x_{(k-1)\cdot m+2}^{k\cdot m}, c_1^m, x_{k\cdot m+1})), x_{k\cdot m+2}^{2km-m}, c_1^m) \overset{(a)}{=}$$

$$= \overset{k}{B}(x_1, \overset{k-1}{B}(x_2^{(k-1)\cdot m+1}, B(x_{(k-1)\cdot m+2}^{k\cdot m+1}, c_1^m)), x_{k\cdot m+2}^{2km-m}, c_1^m) \overset{2.4}{=}$$

$$= \overset{k}{B}(x_1, \overset{k}{B}(x_2^{k\cdot m+1}, c_1^m), x_{k\cdot m+2}^{2km-m}, c_1^m) \overset{(b)}{=}$$

$$= A(x_1, A(x_2^{k\cdot m+1}), x_{k\cdot m+2}^{2km-m}).$$

Sketch of the proof of $\overset{\circ}{2}$*:*

$$A(x_1^m, b_1^{(k-1)\cdot m}) = b_{(k-1)\cdot m+1}^{k\cdot m} \overset{(b)}{\Leftrightarrow} \overset{k}{B}(x_1^m, b_1^{(k-1)\cdot m}, c_1^m) = b_{(k-1)\cdot m+1}^{k\cdot m} \overset{2.5}{\Leftrightarrow}$$

$$B(x_1^m, (\overset{k-1}{B} b_1^{(k-1)\cdot m}, c_1^m)) = b_{(k-1)\cdot m+1}^{k\cdot m}.$$

Sketch of the proof of $\overset{\circ}{3}$*:* By $\overset{\circ}{1}, \overset{\circ}{2}$ and by Prop. 2.1.

Sketch of the proof of $\overset{\circ}{4}$*:*

$$A(b_1^{(k-1)\cdot m}, y_1^m) = b_{(k-1)\cdot m+1}^{k\cdot m} \overset{(b)}{\Leftrightarrow} \overset{k}{B}(b_1^{(k-1)\cdot m}, y_1^m, c_1^m) = b_{(k-1)\cdot m+1}^{k\cdot m} \overset{2.4}{\Leftrightarrow}$$

$$\overset{k-1}{B}(b_1^{(k-1)\cdot m}, B(y_1^m, c_1^m)) = b_{(k-1)\cdot m+1}^{k\cdot m} \overset{(a)i=0}{\Longleftrightarrow}$$

$$\overset{k-1}{B}(b_1^{(k-1)\cdot m}, B(c_1^m, y_1^m)) = b_{(k-1)\cdot m+1}^{k\cdot m} \overset{2.4}{\Longleftrightarrow}$$

$$\overset{k}{B}(b_1^{(k-1)\cdot m}, c_1^m, y_1^m) = b_{(k-1)\cdot m+1}^{k\cdot m} \overset{2.3}{\Longleftrightarrow} B(\overset{k-1}{B}(b_1^{(k-1)\cdot m}, c_1^m), y_1^m) = b_{(k-1)\cdot m+1}^{k\cdot m}.$$

Sketch of a part of the proof of $\overset{\circ}{5}$ *[to case $k=4$]:*

$$A(x_1^{2m}, \overset{m}{e}, (c_1^m)^{-1}) \overset{(b)}{=} \overset{4}{B}(x_1^{2m}, \overset{m}{e}, (c_1^m)^{-1}, c_1^m) \overset{2.4}{=}$$

$$= \overset{3}{B}(x_1^{2m}, \overset{m}{e}, B((c_1^m)^{-1}, c_1^m)) \overset{(d)}{=} \overset{3}{B}(x_1^{2m}, \overset{m}{e}, \overset{m}{e}) \overset{2.4}{=}$$

$$= \overset{2}{B}(x_1^m, B(x_{m+1}^{2m}, \overset{m}{e}), \overset{m}{e}) \overset{(a)}{=} \overset{2}{B}(x_1^m, B(x_{m+1}^{2m-1}, \overset{m}{e}, x_{2m}), \overset{m}{e}) \overset{2.4}{=}$$

$$= \overset{2}{B}(x_1^{2m-1}, e, B(\overset{m-1}{e}, x_{2m}, \overset{m}{e})) \overset{(a)}{=} \overset{2}{B}(x_1^{2m-1}, e, B(\overset{m-1}{e}, \overset{m}{e}, x_{2m})) \overset{2.4}{=}$$

$$= \overset{3}{B}(x_1^{2m-1}, \overset{m}{e}, \overset{m}{e}, x_{2m}) \overset{(d)}{=} \overset{3}{B}(x_1^{2m-1}, \overset{m}{e}, B((c_1^m)^{-1}, c_1^m), x_{2m}) = {}^{2}$$

$$= \overset{3}{B}(x_1^{2m-1}, \overset{m}{e}, B((\bar{c}_1^m, c_1^m), x_{2m}) \overset{2.4}{=} \overset{3}{B}(x_1^{2m-1}, \overset{m}{e}, \bar{c}_1, B((\bar{c}_2^m, c_1^m, x_{2m})) \overset{(a)}{=}$$

$$= \overset{3}{B}(x_1^{2m-1}, \overset{m}{e}, \bar{c}_1, B(\bar{c}_2^m, x_{2m}, c_1^m)) \overset{2.4}{=} \overset{4}{B}(x_1^{2m-1}, \overset{m}{e}, \bar{c}_1^m, \bar{c}_2^m, x_{2m}, c_1^m) =$$

$$= \overset{4}{B}(x_1^{2m-1}, \overset{m}{e}, (c_1^m)^{-1}, x_{2m}, c_1^m) \overset{(b)}{=} A(x_1^{2m-1}, \overset{m}{e}, (c_1^m)^{-1}, x_{2m}).$$

[2] $\bar{c}_1^m = (c_1^m)^{-1}$

By $\overset{\circ}{2} - \overset{\circ}{4}$, Prop. 2.2 and by $\overset{\circ}{5}$, we obtain $(Q; A)$ is a $(km, m)-$group with condition (c). □

Remark 3.1. a) In [3] the following proposition is proved. Let $(Q; A)$ be a $(k \cdot m, m)-$group, $m \geq 2$, $k \geq 3$ and let

$$\mathbf{A}(x_1^m, x_{m+1}^{2m}, \ldots, x_{(k-1)\cdot m+1}^{k\cdot m}) \overset{def}{=} A(x_1^{k\cdot m})$$

for all $x_1^{k\cdot m} \in Q$. Then there exist binary group $(Q^m, \mathbf{B})$, an element $c_1^m \in Q^m$ and an automorphism φ of this group, such that for each $x_1^m, x_{m+1}^{2m}, \ldots, x_{(k-1)\cdot m+1}^{k\cdot m} \in Q^m$

$$\mathbf{A}(x_1^m, x_{m+1}^{2m}, \ldots, x_{(k-1)\cdot m+1}^{k\cdot m}) = \overset{k}{\mathbf{B}}(x_1^m, \varphi(x_{m+1}^{2m}), \ldots, \varphi^{k-1}(x_{(k-1)\cdot m+1}^{k\cdot m}), c_1^m),$$

$$\varphi(c_1^m) = c_1^m$$

and

$$\mathbf{B}(\varphi^{k-1}(x_1^m), c_1^m) = \mathbf{B}(c_1^m, x_1^m).$$

b) $\mathbf{B}, \varphi$ and c_1^m from a), according to [7], are defined in the following way

$$\mathbf{B}(x_1^m, y_1^m) \overset{def}{=} A(x_1^m, a_1^{(k-2)\cdot m}, y_1^m),$$

$$\varphi(x_1^m) \overset{def}{=} A(\mathbf{e}(a_1^{(k-2)\cdot m}), x_1^m, a_1^{(k-2)\cdot m})$$

and

$$c_1^m \overset{def}{=} A\Big(\overbrace{\mathbf{e}(a_1^{(k-2)\cdot m})}^{k}\Big)$$

for all $x_1^m, y_1^m \in Q^m$, where $(Q; A)$ is a $(k \cdot m, m)-$group, $\mathbf{e}$ its $\{1, n-m+1\}-$neutral operation and $k \geq 3$. *[*Cf. Th. 3.1-IV in [9]*]*

c) If condition (c) from Th. 3.2 in $(Q; A)$ holds, then $\varphi(x_1^m) = x_1^m$ for all $x_1^m \in Q^m$.

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